Lecture #8: Low density parity check codes
Outline of the lecture

- Introduction
- Tanner graphs
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- Tanner graphs
- Construction of regular LDPC codes

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Error control coding: An introduction to linear block codes
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- Introduction
- Tanner graphs
- Construction of regular LDPC codes
  - Gallager’s construction
  - MacKay’s construction
- Irregular LDPC codes
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- Construction of regular LDPC codes
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  - MacKay’s construction
- Irregular LDPC codes
- Random construction of irregular LDPC codes.

Definitions

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- A source is **sparse** if its density is less than 0.5.
- A vector \( \mathbf{v} \) is **very sparse** if its density vanishes as its length increases.
- The **overlap** between two vectors is the number of 1’s in common between them.

Low-density parity check codes

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- A regular $(n, w_c, w_r)$ LDPC code is a code of blocklength $n$ with a $m \times n$ parity check matrix where each column contains a small fixed number, $w_c \geq 3$, of 1's and each row contains a small fixed number, $w_r \geq w_c$, of 1's.

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- In other words,
  - Each parity check constraint involves $w_r$ codebits, and each codebit is involved in $w_c$ constraints.

Low-density implies that $w_c \ll m$ and $w_r \ll n$. 
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- In other words,
  - Each parity check constraint involves \( w_r \) codebits, and each codebit is involved in \( w_c \) constraints.
  - Low-density implies that \( w_c << m \) and \( w_r << n \).
  - Number of ones in the parity check matrix \( H = w_c \cdot n = w_r \cdot m \).

\[ m \geq n - k \implies R = k/n \geq 1 - (w_c/w_r), \text{ and thus } w_c < w_r. \]
Regular low-density parity check code

Example of a regular low density code matrix; \( n = 20, \ w_c = 3, \ w_r = 4 \)

| 1 | 1 | 1 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 0 | 0 | 0 | 0 | 1 | 1 | 1 | 1 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 0 | 0 | 0 | 0 | 0 | 0 | 1 | 1 | 1 | 1 | 0 | 0 | 0 | 0 | 0 | 1 | 1 | 1 | 1 | 0 |
| 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 1 | 1 | 1 | 0 | 0 | 0 | 0 | 0 | 0 |
| 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 1 | 1 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 1 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 |
| 0 | 1 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 1 | 0 |
| 0 | 0 | 0 | 1 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 1 | 0 |
| 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 1 |
| 1 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 1 |
| 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 1 | 0 |
| 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 1 | 0 |

A bipartite graph is one in which the nodes can be partitioned into two classes, and no edge can connect nodes from the same class.
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A Tanner graph for an LDPC code is a bipartite graph such that:

- One class of nodes is the “variable nodes” corresponding to \( n \) bits in the codeword.
A *bipartite graph* is one in which the nodes can be partitioned into two classes, and no edge can connect nodes from the same class.

A *Tanner graph* for an LDPC code is a bipartite graph such that:
- One class of nodes is the “variable nodes” corresponding to \( n \) bits in the codeword.
- Second class of nodes is “check nodes” corresponding to \( m \) parity check equations.

An edge connects a variable node to the check node if and only if that particular bit is included in the parity check equation.
Example of a regular low density code matrix; \( n = 12, w_c = 3, w_r = 6 \)

\[
H = \begin{pmatrix}
1 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 0 \\
1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 1 & 1 & 1 \\
1 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 1 & 0 & 1 & 1 \\
0 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 1 & 0
\end{pmatrix}
\]

A cycle of length \( l \) in a Tanner graph is a path comprised of \( l \) edges from a node back to the same node.
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Example: The bipartite graph has a cycle of length six.
Tanner Graphs

- The length of smallest cycle in the graph is known as its girth.

- When decoding LDPC codes using sum-product algorithm, the number of independent iterations of the algorithm is proportional to the girth of its associated Tanner graph.
**Tanner Graphs**

- The length of smallest cycle in the graph is known as its girth.
- When decoding LDPC codes using sum-product algorithm, the number of independent iterations of the algorithm is proportional to the girth of its associated Tanner graph.
- The girth of this Tanner graph is six.

**Gallager’s construction for regular \((n, w_c, w_r)\) code**

- Let, \(n\) be the transmitted block-length of an information sequence of length \(k\). \(m\) is the number of parity check equations.
Gallager’s construction for regular \((n, w_c, w_r)\) code

- Let, \(n\) be the transmitted block-length of an information sequence of length \(k\). \(m\) is the number of parity check equations.
- Construct a \(m \times n\) matrix with \(w_c\) 1’s per column and \(w_r\) 1’s per row. (An \((n, w_c, w_r)\) code)
- Divide a \(m \times n\) matrix into \(w_c \frac{m}{w_c} \times n\) sub-matrices, each containing a single 1 in each column.
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Divide a \(m \times n\) matrix into \(w_c \frac{m}{w_c} \times n\) sub-matrices, each containing a single 1 in each column.

The first of these sub-matrices contains all 1’s in descending order, i.e. the \(i\)’th row contains 1’s in columns \((i - 1) \cdot w_r + 1\) to \(i \cdot w_r\).

The other sub-matrices are merely column permutations of the first sub-matrix.
Gallager’s construction for regular \((n, w_c, w_r)\) code

Example of a regular low density code matrix; \(n = 20\), \(w_c = 3\), \(w_r = 4\)

MacKay’s construction

- An \(m \times n\) matrix (\(m\) rows, \(n\) columns) is created at random with weight per column \(w_c\), and weight per row \(w_r\), and overlap between any two columns no greater than 1.

Reference:
MacKay’s construction

- An \( m \) by \( n \) matrix (\( m \) rows, \( n \) columns) is created at random with weight per column \( w_c \), and weight per row \( w_r \), and overlap between any two columns no greater than 1.
- Another way of constructing regular LDPC codes is to build the parity check matrix from non-overlapping random permutation matrices.

Reference:

Construction of low density parity check codes

- A permutation matrix is just the identity matrix with its row re-ordered, e.g.

\[
P = \begin{bmatrix}
0 & 0 & 0 & 1 & 0 \\
1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 \\
0 & 1 & 0 & 0 & 0 \\
\end{bmatrix}
\]
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0 & 0 & 0 & 0 & 1 \\
0 & 1 & 0 & 0 & 0
\end{bmatrix}
\]

- A circulant matrix is defined by the property that each row is a cyclic shift of the previous row to the right by one position.

\[
C = \begin{bmatrix}
0 & 1 & 0 & 0 & 1 \\
1 & 0 & 1 & 0 & 0 \\
0 & 1 & 0 & 1 & 0 \\
0 & 0 & 1 & 0 & 1 \\
1 & 0 & 0 & 1 & 0
\end{bmatrix}
\]

MacKaye’s construction

![Schematic Illustration of Regular Gallager Codes](image)

**Notation:** An integer represents a number of permutation matrices superposed on the surrounding square.

<table>
<thead>
<tr>
<th>Column Weight</th>
<th>Fraction of columns</th>
<th>Row weight</th>
<th>Fraction</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>1</td>
<td>6</td>
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Irregular low-density parity check codes

- For an irregular low-density parity-check code the degrees of each set of nodes are chosen according to some distribution.

A degree distribution $\gamma(x) = \sum_i \gamma_i x^{i-1}$ is simply a polynomial with nonnegative real coefficients satisfying $\gamma(1) = 1$. 
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- A degree distribution \( \gamma(x) = \sum_i \gamma_i x^{i-1} \) is simply a polynomial with nonnegative real coefficients satisfying \( \gamma(1) = 1 \).
- An irregular low-density code is a code of block-length N with a sparse parity check matrix where column distribution \( \lambda(x) \) and row distribution \( \rho(x) \) is respectively given by

\[
\lambda(x) = \sum_{i>1} \lambda_i x^{i-1}
\]
\[
\rho(x) = \sum_{i>1} \rho_i x^{i-1}
\]

where \( \lambda_i \) and \( \rho_i \) denote the fraction of edges incident to variable and check nodes with degree \( i \), respectively.

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Irregular Low-density parity check code

\[
\lambda(x) = \frac{1}{4} x + \frac{7}{12} x^2 + \frac{1}{12} x^3 + \frac{1}{12} x^4
\]
\[
\rho(x) = \frac{1}{3} x^4 + \frac{1}{3} x^5 + \frac{1}{3} x^6
\]
Construction of irregular LDPC code

**Step 1:** Selecting a profile that describes the desired number of columns of each weight and the desired number of rows of each weight.

**Step 2:** Construction method, i.e. algorithm for putting edges between the vertices in a way that satisfies the constraints.

Random construction of Irregular LDPC Codes

The edges are placed “completely at random” subject to the profile constraints. One way of implementing it is shown below.

- Make a list of all columns in the matrix, with each column appearing in the list a number of times equal to its weight.
Random construction of Irregular LDPC Codes

The edges are placed “completely at random” subject to the profile constraints. One way of implementing it is shown below.

- Make a list of all columns in the matrix, with each column appearing in the list a number of times equal to its weight.
- Make a similar list of all rows in the matrix, with each row appearing in the list a number of times equal to its weight.
- Map one list onto the other by a random permutation, taking care not to create duplicate entries.
Construction of Irregular LDPC Codes

Notation: integers “3” and “9” represent the column weights.

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</tr>
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